Math 1552

Sections 6.1 and 6.2: Volumes of Revolution

Math 1552 lecture slides adapted from the course materials
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Quiz 5 is on Thursday July 22, 2021 (during the last 25 minutes of the studio session).

Topics List:

- power series
- radius of convergence and IC of power series
- Taylor polynomials
- Taylor series
- Taylor series and remainder terms + error bounds in approximating series (see lecture notes)

Notes on how to compute $I = \int_0^1 (\sin^{-1}(y))^2 \, dy$.

$\to$ IBP once:

$$\left(\begin{array}{ll} u = \sin^{-1}(y)^2 & dv = dy \\ du = \frac{2\sin^{-1}(y)}{\sqrt{1-y^2}} dy & v = y \end{array} \right)$$

$$\int_0^1 u \, dv = uv\Big|_0^1 - \int_0^1 v \, du$$

$$I = y \sin^{-1}(y)^2 \Big|_0^1 - 2\int_0^1 \frac{y}{\sqrt{1-y^2}} \sin^{-1}(y) \, dy$$

(1) $\int \frac{y}{\sqrt{1-y^2}} = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{1-y^2} + C$

$u = 1-y^2$, $du = -2y$

(2) IBP: $u = \sin^{-1}(y)$, $du = \frac{dy}{\sqrt{1-y^2}}$, $dv = \frac{y\,dy}{\sqrt{1-y^2}}$, $v = -\sqrt{1-y^2}$

$$I = \left(\frac{\pi}{2}\right)^2 + \underbrace{2\sin^{-1}(y)\sqrt{1-y^2}}_{=0}\Big|_0^1 - 2\int_0^1 dy$$

$$= \frac{\pi^2}{4} - 2 \quad (**)$$

So $V = \pi\int_0^1 \left[\frac{\pi^2}{4} - \left(\sin^{-1}(y)\right)^2\right] dy$

by (**)

$$= \pi\left[\frac{\pi^2}{4} - \left(\frac{\pi^2}{4} - 2\right)\right] = 2\pi$$

Volumes by Cylindrical Shells (Section 6.2)

We can find the volume of the solid generated by revolving the region bounded by $y=f(x)$, $x=a$, $x=b$, and the x-axis using the basic formulas:

$$V = 2\pi \int_a^b x[f(x)]dx \quad \text{(revolved about } y\text{-axis)}$$

$$V = 2\pi \int_a^b y[g(y)]dy \quad \text{(revolved about } x\text{-axis)}$$

Notes about the Shell Method:

- In the shell method, the variable of integration is the *opposite* of the axis of revolution.
- To use the washer method with shells:

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx = 2\pi \int_a^b x[\overset{(x)}{top} - \overset{(at\ x)}{bottom}]dx$$

OR

$$V = 2\pi \int_a^b y[f(y) - g(y)]dy = 2\pi \int_a^b y[\overset{(at\ y)}{right} - \overset{(at\ y)}{left}]dy$$

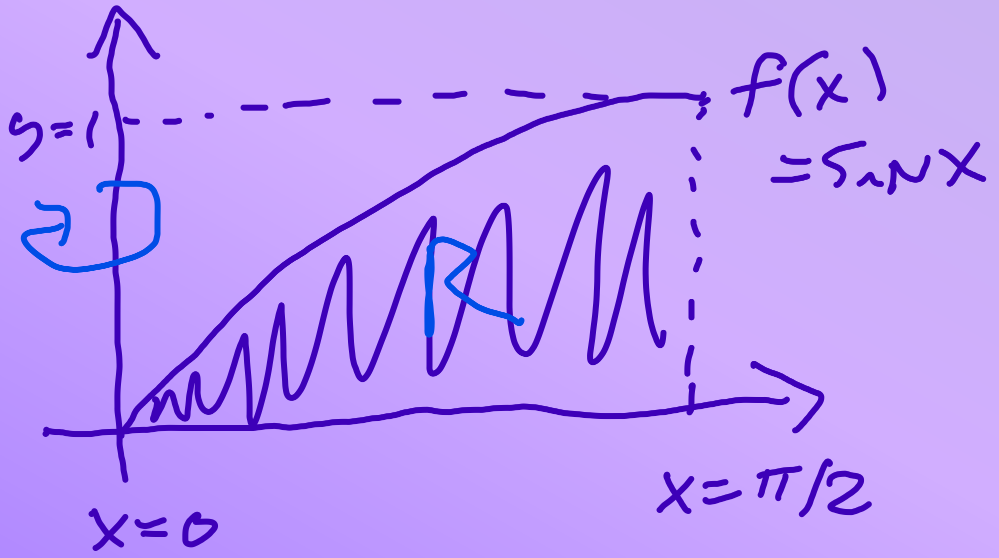
Example 4:

Find the volume of the solid generated by revolving the region bounded by the curves:

$y = \sin x$, the x-axis, and the lines

$= f(x)$

$x = 0,\ x = \dfrac{\pi}{2}$ about the y-axis.



→ use the shell method:

$$V = 2\pi \int_0^{\pi/2} x \cdot \sin(x)\,dx \quad (\text{use IBP})$$

$$\begin{pmatrix} u = x & dv = \sin(x)\,dx \\ du = dx & v = -\cos(x) \end{pmatrix}$$

Ex 4 (cont):

$$V = 2\pi \int_0^{\pi/2} x \cdot \sin x \, dx \quad \text{(apply IBP)}$$

$$= 2\pi \left[uv \Big|_0^{\pi/2} - \int_0^{\pi/2} v \, du \right]$$

$$= 2\pi \left[-x \cdot \cos(x) \Big|_0^{\pi/2} + \int_0^{\pi/2} \cos(x) \, dx \right]$$

0

$$= 2\pi (\sin x) \Big|_0^{\pi/2} = 2\pi (1 - 0) = 2\pi$$

<u>Example</u>: Set up the integral to find the volume bounded by

$y_1 = x + 2$ and $y_2 = x^2$

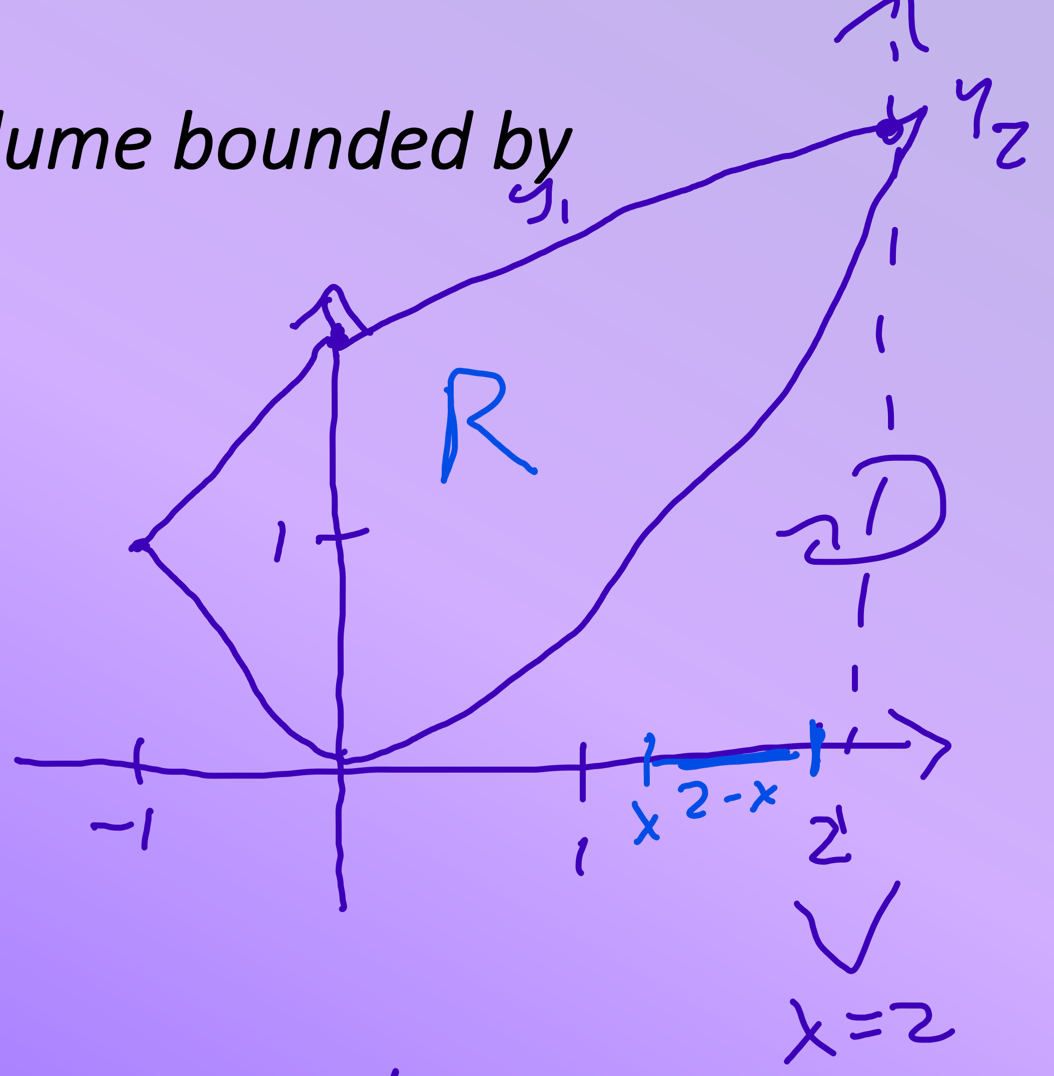
about the line $x = 2$.

(A) $V = 2\pi \int_1^4 (y+2)\left[(y-2) - \sqrt{y}\right] dy$

(B) $V = 2\pi \int_{-1}^2 (x-2)\left[(x+2) - x^2\right] dx$

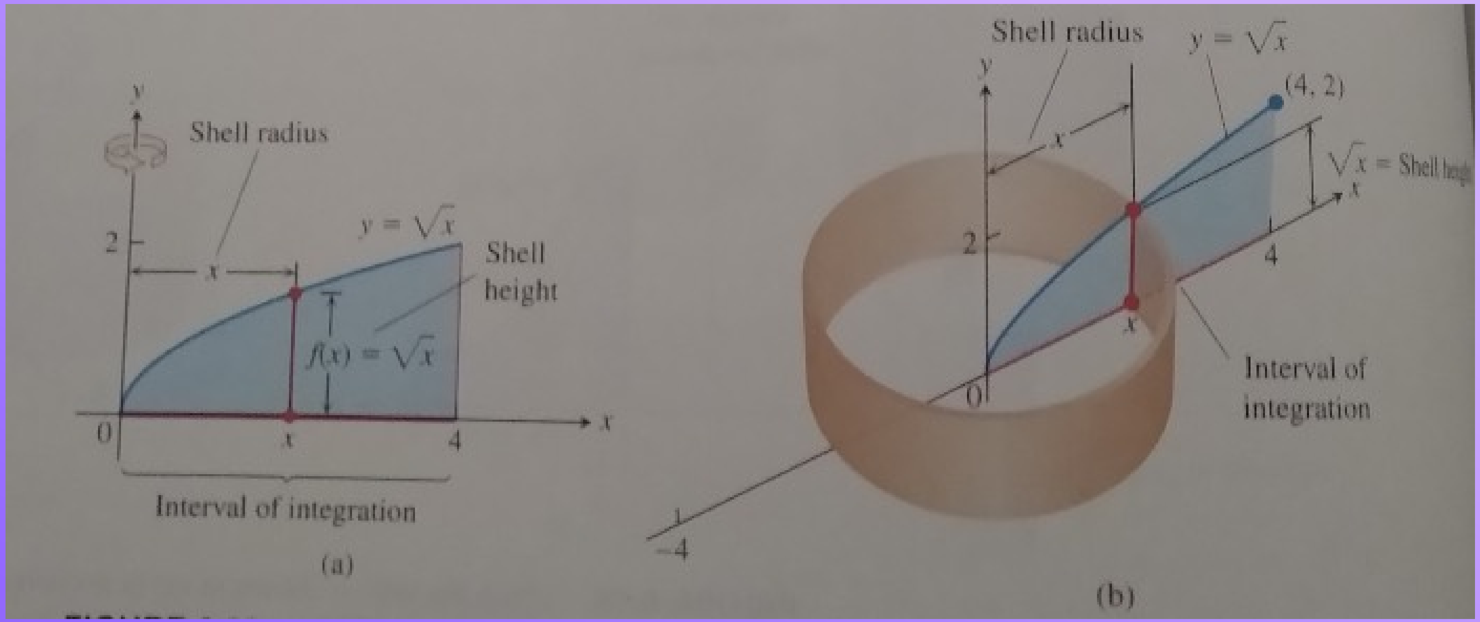
(C) $V = 2\pi \int_1^4 (2-y)\left[(y-2) - \sqrt{y}\right] dy$

$\boxed{\text{(D)}}$ $V = 2\pi \int_{-1}^2 (2-x)\left[(x+2) - x^2\right] dx$



(with the shell method
→ come back with illustration on Friday)

Example of the method:



$$V = \int_a^b 2\pi \underbrace{(\text{shell radius at } x)}_{2-x} \underbrace{(\text{shell height at } x)}_{\text{top - bottom}} dx$$

→ find the intersection points: $(y_1 = y_2)$

$$x^2 - x - 2 = 0 \Longleftrightarrow (x-2)(x+1) = 0$$

$$\Longleftrightarrow x = 2, -1$$

→ using the shell method, since we are revolving around a line parallel to the y-axis, the variable of integration is x

→ for all $-1 \leq x \leq 2$, $y_1(x) \geq y_2(x)$

→ $$V = 2\pi \int_{-1}^{2} \Big(\overbrace{\text{distance from } x=2 \text{ to } x}^{2-x}\Big)\Big(\overset{y_1(x)}{\text{top}} - \overset{y_2(x)}{\text{bottom}}\Big)\,dx$$

$$= 2\pi \int_{-1}^{2} (2-x)(x+2-x^2)\,dx$$

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Extra Hints and Problem Solutions on Volumes of Revolution

Example A:

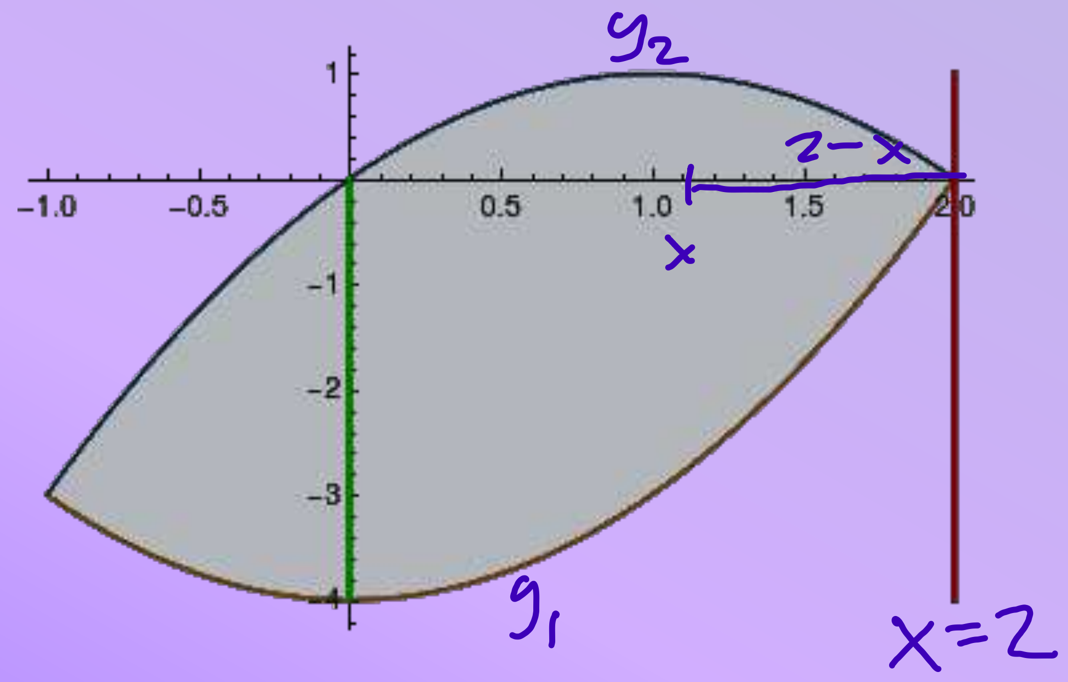
Find the volume of the solid generated by revolving the region bounded by the curves

$$y_1(x) = x^2 - 4 \qquad (\textit{in orange})$$

AND

$$y_2(x) = 2x - x^2 \qquad (\textit{in blue})$$

around the line x=2.



Use the SHELL METHOD (since we are revolving about a vertical line):

$$V = 2\pi \times \int_a^b (\textbf{distance to line at x}) \times (\underbrace{\textbf{height of region at x}}_{\text{use the washer method with shells}})\, dx$$

$$= 2\pi \times \int_{-1}^{2} (2-x)(\underbrace{4+2x-2x^2}_{y_2(x)-y_1(x) \equiv \text{top} - \text{bottom}})dx \quad (*)$$

$$= 27\pi$$

→ Scratchwork:

$(2-x)(4+2x-2x^2)$

$= 8 + \cancel{4x} - 4x^2 - \cancel{4x} - 2x^2 + 2x^3$

$= 8 - 6x^2 + 2x^3 = 2(4 - 3x^2 + x^3)$

→ so working from (*):

$$V = 4\pi \int_{-1}^{2} (4 - 3x^2 + x^3)\,dx = 4\pi \left(4x - x^3 + \frac{x^4}{4}\right)\Bigg|_{-1}^{2}$$

$$= 27\pi$$

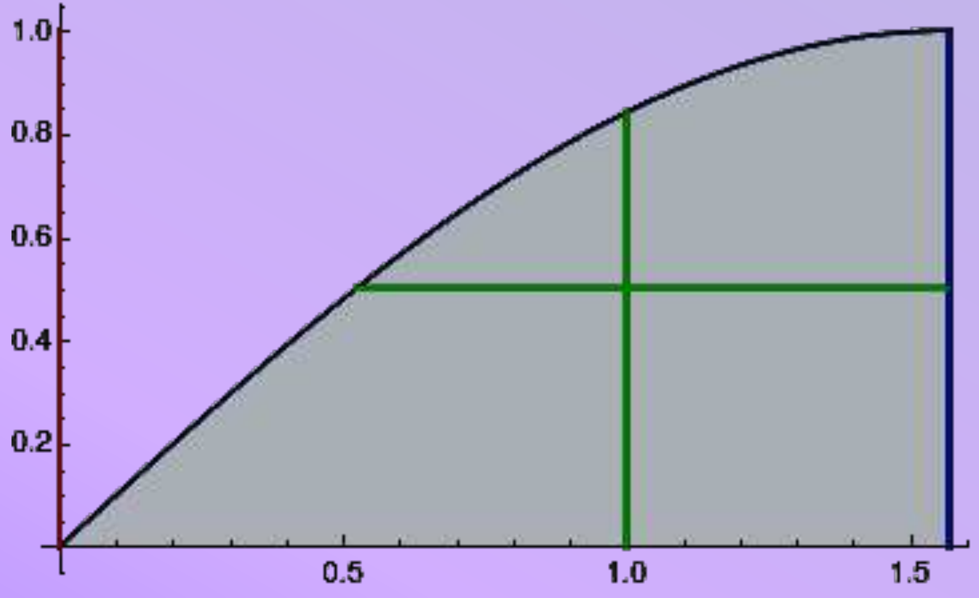
Example C:

Find the volume of the solid generated by revolving the region bounded by the curve

$$y = \sin(x)$$

and the x-axis and the lines $x = 0, \dfrac{\pi}{2}$

about the y-axis.



we did this already today

SHELL METHOD SETUP (Vertical Slices): $V = 2\pi \times \displaystyle\int_0^{\frac{\pi}{2}} x\sin(x)dx$

WASHER METHOD SETUP (Horizontal Slices): $V = \pi \times \displaystyle\int_0^1 \left[\frac{\pi^2}{4} - \left(\sin^{-1}(y)\right)^2\right] dx$